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LETTER TO THE EDITOR

Integrable model of interacting XY chains*

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Abstract. An exact diagonalization of the Hamiltonian for the one-dimensional quantum model consisting of an arbitrary number of isotropic XY chains connected by many-spin interactions is carried out with the help of the generalized nested Bethe ansatz. An exact solution for the ground-state energy of the considered model is obtained.

Considerable progress in the construction and investigation of new integrable models in statistical mechanics has been achieved in the last two decades [1–3]. First of all it is connected with the understanding of the key role of the Yang–Baxter, or star-triangle equations [1–3, 4] in the study of integrable statistical systems. Any solution of these equations provides us with an integrable spin model both in classical and quantum statistical mechanics. Most of the new integrable models have been constructed in this way. These models present the generalization of the classical integrable systems considered by Bethe, Yang and Baxter for the case of an arbitrary spin $S>\frac{1}{2}$; see e.g. [5] and references therein. It should be noted that the continuous integrable models in quantum field theory are solved by the further development of Bethe's ideas in the papers of Yang [4] and Sutherland [6] which can be considered as complicated systems with four-fermion interactions between subsystems [7–9]. In the lattice case there is only one example of such type—the one-dimensional Hubbard model [10], which can be presented as two isotropic XY chains connected by ZZ interactions. Up to now integrable generalizations of this model for the case of an arbitrary number of sublattices have not been constructed. Thus the consideration of other multisublattice systems for construction of new integrable lattice models is of interest.

In a previous paper [11] we considered an integrable system which can be presented as two isotropic XY chains connected by three-spin interactions. In the present letter we consider a generalization of this model for the case of an arbitrary number of chains.

The Hamiltonian of the considered model has the following form:

$$H=-\sum_{j=1}^{L}\left\{\sum_{\alpha=1}^{N}(\sigma^{+}_{j(\alpha)}\sigma^{-}_{j+1(\alpha)}+\sigma^{+}_{j+1(\alpha)}\sigma^{-}_{j(\alpha)})\times\exp\left[-\eta\sum_{\substack{\alpha'=1\\ \alpha'\neq\alpha}}^{N}\sigma^{+}_{j+\theta(\alpha-\alpha')(\alpha')}\sigma^{-}_{j+\theta(\alpha-\alpha')(\alpha')}\right]\right\} \tag{1}$$

where

$$\theta(\alpha-\alpha')=\begin{cases}1 & \alpha>\alpha'\\ 0 & \alpha<\alpha'\end{cases}$$

$$\sigma^{\pm}_{j(\alpha)}=\tfrac{1}{2}[\sigma^{x}_{j(\alpha)}\pm i\sigma^{y}_{j(\alpha)}]$$

* Dedicated to Professor M E Fisher on the occasion of his 60th birthday.

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are Pauli spin matrices of the jth lattice site on the sublattice α ($\alpha = 1, 2, \ldots, N$). We use the usual periodical boundary conditions ($\sigma^{\tau}_{L+1(\alpha)} = \sigma^{\tau}_{1(\alpha)}$), where L is the number of sites on each of N sublattices. The value η determines the interaction between the chains. Because of various symmetries of the Hamiltonian (1) it is sufficient to consider the model for $\eta > 0$.

By using the Jordan–Wigner transformation [12] the Hamiltonian (1) can be presented in terms of the creation and annihilation operators. Up to the boundary conditions such a Hamiltonian is obtained from (1) by the formal substitution of $a^{*}_{j(\alpha)}$ and $a_{j(\alpha)}$ for $\sigma^{-}_{j(\alpha)}$ and $\sigma^{+}_{j(\alpha)}$, respectively. In this case we have the model of a conductor describing the hopping of electrons on the one-dimensional chains. The probability of an electron jump in any direction is determined by the occupation of neighbouring sites on the other chains. This situation is analogous to that realized in quasi-one-dimensional conductors [13]. Thus one may hope that the considered model will be useful in the study of such systems. In the present letter, however, we consider only the mathematical problem of the diagonalization of the Hamiltonian (1).

To calculate the eigenfunctions of the Hamiltonian (1)

$$\Psi = \sum f(x_1, \alpha_1; \ldots, X_n, \alpha_n)\sigma_{X_1}(\alpha_1) \ldots \sigma_{X_n}(\alpha_n)|0\rangle \tag{2}$$

we use the generalized nested Bethe ansatz [4, 6]. Let us consider that in the expression (2) there are m_1 spins directed $\uparrow$ on the first chain, m_2 spins $\uparrow$ on the second chain and so on. Let us divide the coordinate space into a number of regions characterized by the fixed sequence of the particle coordinates $1 \leqslant X_{Q_1} \leqslant \ldots \leqslant X_{Q_n} \leqslant L$ where Q is a permutation $[Q_1, \ldots, Q_n]$ of the numbers $1, 2, \ldots, n$. Let us seek the amplitude f in each of these regions in the form of a superposition of plane waves with n 'wavenumbers' k_j

$$f(X_1, \alpha_1; \ldots; X_n, \alpha_n) = \sum_{P} (-1)^{P}(-1)^{Q} A^{\alpha_1 \ldots \alpha_n}_{P_1 \ldots P_n} \prod_{j=1}^{n} \exp(\mathrm{i}k_{P_j} X_{Q_j}). \tag{3}$$

The sum is over all permutations $P = [P_1, \ldots, P_n]$ of the numbers $1, 2, \ldots, n$. If no two particle coordinates are equal the ansatz (3) satisfies $H|\Psi\rangle = E|\Psi\rangle$ with energy and momentum

$$E = -2 \sum_{j=1}^{n} \cos k_j \qquad P = \sum_{j=1}^{n} k_j. \tag{4}$$

The requirement that $H|\Psi\rangle = E|\Psi\rangle$ is also satisfied on the boundaries of each region provided the coefficients $A^{\alpha Q}_{P}$ satisfy the following equations:

$$\begin{aligned} A^{\ldots\alpha\beta\ldots}_{\ldots P_1 P_2 \ldots} &= \sum_{\delta,\gamma=1}^{N} S^{\gamma\delta}_{\alpha\beta}(\tfrac{1}{2}(k_{P_2} - k_{P_1}))A^{\ldots\delta\gamma\ldots}_{\ldots P_2 P_1 \ldots} \\ A^{\alpha_1 \ldots \alpha_n}_{P_1 \ldots P_n} &= A^{\alpha_2 \ldots \alpha_n \alpha_1}_{P_2 \ldots P_n P_1} \exp(\mathrm{i}k_{P_1} L) \end{aligned} \tag{5}$$

where the non-vanishing elements of the S-matrix are

$$\begin{aligned} &S^{\alpha\alpha}_{\alpha\alpha}(k) = 1 \qquad S^{\alpha\beta}_{\alpha\beta}(k) = \sin k/\sin(k + \mathrm{i}\eta) \\ &S^{\beta\alpha}_{\alpha\beta}(k) = \mathrm{i}\sinh\eta \exp[\mathrm{i}\,\mathrm{sgn}(\beta - \alpha)k]/\sin(k + \mathrm{i}\eta) \qquad \alpha \neq \beta. \end{aligned} \tag{6}$$

A necessary and sufficient condition for the compatibility of equations (5) is the fulfilment of the Yang–Baxter equations [4, 6]. In our case the S-matrix has a well

known form [14-17] and satisfies these equations, and we may use the quantum method of inverse problems [18, 19] to solve (5). The problem reduces to the diagonalization of the transfer matrix of the non-uniform model of the non-intersecting strings [20]. As a result, we obtain the system of transcendental equations for finding the k_j and additional unknown quantities $\Lambda_\gamma^{(k)}$

$$\begin{aligned}
&k_j L + \sum_{\alpha=1}^{M_{N-1}} \theta(k_j - \Lambda_\alpha^{(1)}; \eta') = 2\pi I_j \qquad (j = 1, 2, \ldots, n) \\
&\sum_{\alpha=1}^{M_{N-k+1}} \theta(\Lambda_\gamma^{(k)} - \Lambda_\alpha^{(k-1)}; \eta') + \sum_{\delta=1}^{N_{N-k-1}} \theta(\Lambda_\gamma^{(k)} - \Lambda_\delta^{(k+1)}; \eta') - \sum_{\gamma'=1}^{M_{N-k}} \theta(\Lambda_\gamma^{(k)} - \Lambda_{\gamma'}^{(k)}; 2\eta') \\
&\qquad = 2\pi J_\gamma^{(N-k)} \qquad (\gamma = 1, 2, \ldots, M_{N-k}) \\
&\qquad k = 1, 2, \ldots, N-1 \qquad (\Lambda_j^{(0)} \equiv k_j) \qquad \eta' = \tfrac{1}{2}\eta
\end{aligned} \tag{7}$$

where

$$\theta(k; \eta) = 2\tan^{-1}(\cot\eta \tan\tfrac{1}{2}k) \qquad -\pi \leqslant \theta(k, \eta) < \pi$$

and I_j and $J_\gamma^{(k)}$ are integer (half-integer) numbers for odd (even) $M_{N-1}+1$ and $M_{k-1}+m_{k+1}$, respectively, and

$$M_k \equiv \sum_{j=1}^{k} m_j \qquad (M_N = n;\ M_0 = 0)$$

is the summary number of spins $\uparrow$ on the first, second, ... and kth sublattices; $\Sigma_{\delta=1}^{M}$ is equal to zero if $M = 0$.

Let us consider the limit $L \to \alpha$ for fixed ratios n/L and m_k/L. In this limit the k_j fill the interval $(-Q, Q)$ uniformly with density $\rho(k)$ and we have

$$\int_{-Q}^{Q} \rho(k)\,\mathrm{d}k = n/L. \tag{8}$$

In the same way we introduce the distribution functions $\sigma_i(\Lambda)$ for the numbers $\Lambda_\beta^{(k)}$ $(\beta = 1, 2, \ldots, M_{N-k}, k = 1, 2, \ldots, N-1)$

$$\int_{-B_k}^{B_k} \sigma_k(\Lambda)\,\mathrm{d}\Lambda = M_{N-k}/L. \tag{9}$$

The ground state corresponds to the following values of I_j and $J_\beta^{(1)}$:

$$I_{j+1} - I_j = 1 \qquad J_{\beta+1}^{(N-k)} - J_\beta^{(N-k)} = 1.$$

As a result instead of (7) we obtain the system of integral equations

$$\begin{aligned}
&2\pi\rho(k) - \int_{-B_1}^{B_1} \theta'(k - \Lambda; \eta')\sigma^{(1)}(\Lambda)\,\mathrm{d}\Lambda = 1 \\
&2\pi\sigma^{(k)}(\Lambda) + \int_{-B_k}^{B_k} \theta'(\Lambda - \Lambda'; 2\eta')\sigma^{(1)}(\Lambda')\,\mathrm{d}\Lambda' \\
&\qquad = \int_{-B_{k-1}}^{B_{k-1}} \theta'(\Lambda - \Lambda'; \eta')\sigma^{(k-1)}(\Lambda')\,\mathrm{d}\Lambda' + \int_{-B_{k+1}}^{B_{k+1}} \theta'(\Lambda - \Lambda'; \eta')\sigma^{(k+1)}(\Lambda')\,\mathrm{d}\Lambda' \\
&\qquad (k = 1, 2, \ldots, N-1,) \qquad [B_0 = Q,\ B_N = 0,\ \sigma^{(0)}(\Lambda) = \rho(\Lambda)].
\end{aligned} \tag{10}$$

The expression for the energy (4) has the following form:

$$\frac{1}{L}E = -2\int_{-Q}^{Q} \cos k\rho(k)\,\mathrm{d}k. \tag{11}$$

In the continuum limit when the lattice constant tends to zero the Bethe ansatz equations will have the same form as those of the one-dimensional electron gas [4, 6]. In this sense the model under consideration is the discrete version of this famous integrable system. Certainly the lattice model has a symmetry other than its continuous analogue, and it is manifested in its physical properties. In particular the lattice model, as will be shown later, has non-zero magnetization.

If the limits of integration in (10) on some variable are $(-\pi, \pi)$, then after integration of the corresponding equation we have $m_k = m_{k+1}$. In the most interesting—symmetrical—case we have the same number of spins $\uparrow$ on each chain all $B_k = \pi$. In this case using the Fourier transformation the system of equations (10) is reduced to one integral equation for the unknown function $\rho(k)$

$$2\pi\rho(k) - \int_{-Q}^{Q} \varphi(k-k')\rho(k')\,\mathrm{d}k = 1 \tag{12}$$

where

$$\varphi(k) = 1 - \frac{1}{N} + 2\sum_{n=1}^{\infty} \exp(-n\eta)\sinh[n\eta(N-1)]\cos(nk)/\sinh(n\eta N).$$

The equations (11) and (12) determine the ground-state energy as a function of the magnetization

$$y = 1 - \frac{2}{N}\int_{-Q}^{Q} \rho(k)\,\mathrm{d}k. \tag{13}$$

The solution of the equation (12) can be obtained with the help of numerical integration or using perturbation theory. In particular at large η we have

$$\begin{aligned} \frac{1}{L}E &= -2\frac{\sin Q}{\pi - Qt}\left[1 + \frac{1}{\pi}\left(\frac{2t\sin^2 Q}{\pi - Qt} + Q + \frac{1}{2}\sin 2Q\right)\mathrm{e}^{-2\eta} + \mathrm{O}(\mathrm{e}^{-4\eta})\right] \\ y &= 1 - \frac{2}{N}\left[\frac{Q}{\pi - Qt} + 2\frac{\sin^2 Q}{(\pi - Qt)^2}\mathrm{e}^{-2\eta} + \mathrm{O}(\mathrm{e}^{-4\eta})\right] \end{aligned} \tag{14}$$

where $t = (N-1)/N$. It is clear from these expressions that the considered model has finite magnetization in the zero external field.

In conclusion we shall identify a number of problems arising in connection with the solution of the considered model: the construction of the two-dimensional lattice model of the classical statistical mechanics transfer matrix which will be the generating function of the motion integrals of the Hamiltonian (1); the consideration of the generalizations of the considered model for the case of anisotropic XY chains and, finally, the construction of the integrable model S-matrix which would generalize (6) in the sense of Belavin [17]. The author hopes to consider these problems in further work.

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